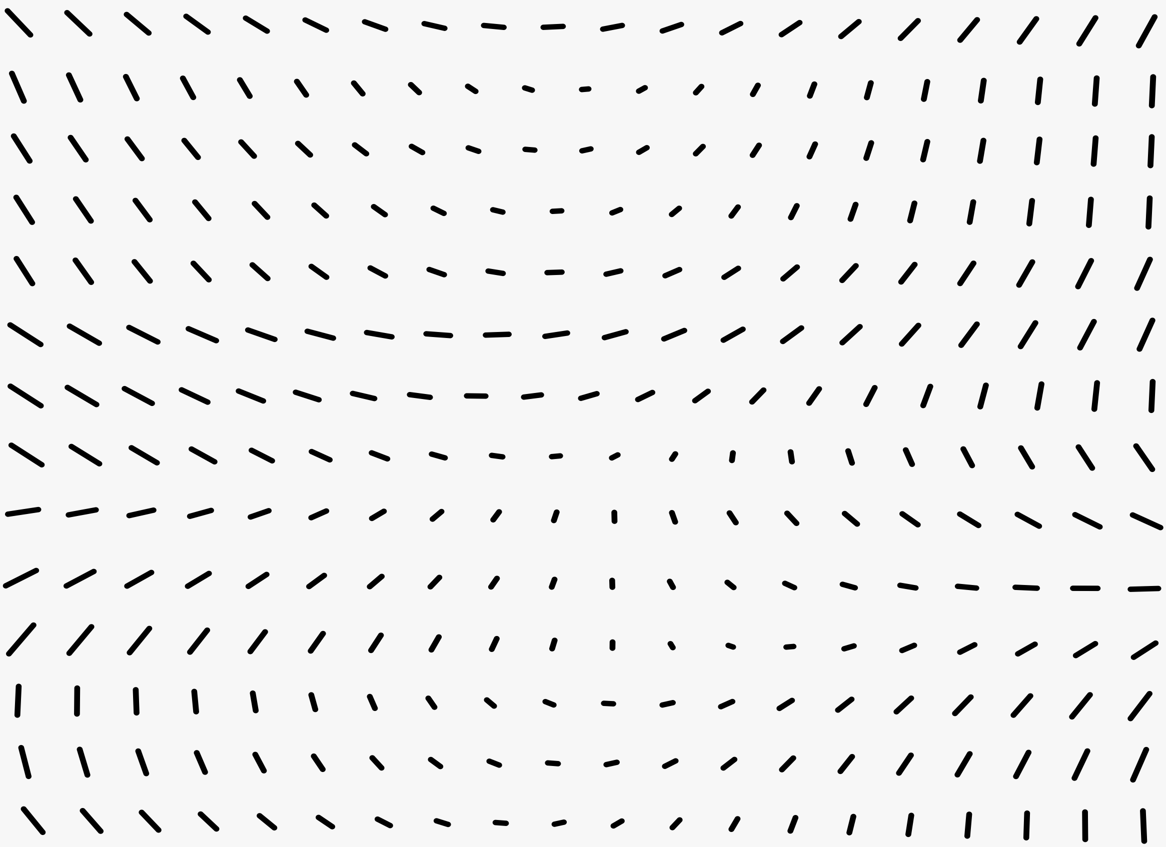
UNIT - 2

CIB SOLUTIONS

ENGINEERING MATHEMATICS - II



Gamma and Beta Functions

1. compute

(a) $\Gamma(4.5)$

$$\Gamma(4.5) = 3.5\,\Gamma(3.5)$$
$$= 3.5 \times 2.5 \times 1.5 \times 0.5\ \Gamma(0.5)$$
$$= 6.5625\sqrt{\pi}$$
$$= 11.63$$

(b) $\Gamma(-3.5)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(-3.5) = \frac{\Gamma(0.5)}{-3.5 \times -2.5 \times -1.5 \times -0.5} = \frac{\sqrt{\pi}}{6.5625}$$
$$= 0.27$$

(c) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin \pi/4} = \frac{\pi}{1/\sqrt{2}} = 4.44$

(d) $\beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} = \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{3!}$

$$= \frac{3}{8} \times \frac{\pi}{6} = \frac{\pi}{16} = 0.196$$

II. Evaluate the following integrals

1) $\int_0^\infty x^7 e^{-x} dx = \Gamma(8) = 7! = 5040$

2) $\int_0^\infty x^3 e^{-4x} dx$ $\quad 4x = y \Rightarrow 4\,dx = dy \Rightarrow \frac{dy}{4} = dx$

$$\int_0^\infty \left(\frac{y}{4}\right)^3 e^{-y} \frac{dy}{4} = \frac{1}{4^4}\int_0^\infty e^{-y} y^3 dy = \frac{1}{4^4}\Gamma(4) = \frac{6}{4\times 4^3} = \frac{3}{128}$$

3) $\int_0^\infty x^{1/2} e^{-x^2} dx$ $\quad t = x^2 \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{dt}{2\sqrt{t}}$

$$\int_0^\infty \frac{t^{1/4} e^{-t} dt}{2t^{1/2}} = \frac{1}{2}\int_0^\infty t^{-1/4} e^{-t} dt = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

4) $\int_0^1 x^5 (1-x)^4 dx = \beta(6,5) = \frac{\Gamma(6)\Gamma(5)}{\Gamma(11)} = \frac{1}{1260}$

5) $\int_0^3 \frac{x^3 dx}{\sqrt{3-x}} = \frac{1}{\sqrt{3}}\int_0^3 \frac{x^3 dx}{\sqrt{1-\frac{x}{3}}}$ $\quad y = \frac{x}{3} \Rightarrow x = 3y$, $dx = 3dy$

$$= \frac{1}{\sqrt{3}}\int_0^1 \frac{3\cdot 3^3 y^3 dy}{\sqrt{1-y}} = \frac{1}{\sqrt{3}}\cdot 3^4 \int_0^1 y^3 (1-y)^{-1/2} dy = 27\sqrt{3}\,\beta(4, 1/2)$$

$$= \frac{27\sqrt{3}\,\Gamma(4)\Gamma(1/2)}{\Gamma(4.5)} = \frac{27\sqrt{3}\cdot 3!\,\cancel{\Gamma(1/2)}}{3.5\times 2.5\times 1.5\times 0.5\,\cancel{\Gamma(1/2)}} = 42.76$$

6) $\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta = \beta(3, \tfrac{5}{2}) \times \tfrac{1}{2} = \tfrac{1}{2} \dfrac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)}$

$= \dfrac{1}{\cancel{2}} \dfrac{\cancel{\times 2!} \times \cancel{1.5} \times \cancel{0.5} \times \cancel{\Gamma(0.5)}}{4.5 \times 3.5 \times 2.5 \times \cancel{1.5} \times \cancel{0.5} \times \cancel{\Gamma(0.5)}} = \dfrac{8}{315} = 0.0254$

7) $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} (\sin\theta)^{1/2} (\cos\theta)^{-1/2} d\theta = \tfrac{1}{2} \beta\left(\tfrac{3}{4}, \tfrac{1}{4}\right)$

$= \tfrac{1}{2} \Gamma(3/4)\Gamma(1/4) = \tfrac{1}{2} \dfrac{\pi}{\sin \pi/4} = \dfrac{\pi}{\sqrt{2}} = 2.22$

8) $\int_0^1 \dfrac{dx}{\sqrt{-x \ln x}} = \int_0^1 \dfrac{dx}{\sqrt{x \ln(\frac{1}{x})}}$

$t = \ln(1/x) \Rightarrow e^{-t} = x$

$dx = -e^{-t} dt$

$t : \infty$ to 0

$= \int_\infty^0 \dfrac{-e^{-t} dt}{\sqrt{e^{-t} t}} = \int_0^\infty e^{-t/2} t^{-1/2} dt$

$y = \frac{t}{2} \Rightarrow t = 2y$

$dt = 2dy$

$= \int_0^\infty e^{-y} \frac{1}{\sqrt{2}} y^{-1/2} \, 2dy = \sqrt{2} \int_0^\infty e^{-y} y^{-1/2} dy = \sqrt{2}\, \Gamma(1/2) = \sqrt{2}\sqrt{\pi}$

$= 2.5066$

9) $\int_0^\infty \sqrt{y}\, e^{-y^2} dy = \int_0^\infty e^{-y^2} y^{1/2} dy = \tfrac{1}{2} \Gamma(\tfrac{3}{4})$

10) $\int_0^1 x^4 \left[\ln\left(\frac{1}{x}\right)\right]^3 dx$

$\ln(1/x) = t \Rightarrow x = e^{-t}$

$dx = -e^{t} dt$

$$= \int_0^\infty e^{-4t}\, t^3\, e^{-t}\, dt = \int_0^\infty e^{-5t} t^3\, dt$$

$y = 5t$

$\frac{dy}{5} = dt$

$$= \int_0^\infty e^{-y} \left(\frac{y}{5}\right)^3 \frac{dy}{5} = \frac{1}{625} \int_0^\infty e^{-y} y^3\, dy = \frac{\Gamma(4)}{625} = \frac{6}{625}$$

Bessel Functions

1. Obtain series solution of Bessel's DE

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

(n is a real constant)

Assume $y = \sum_{r=0}^\infty a_r x^{r+k}$

$$y' = \sum_{r=0}^\infty (r+k) a_r\, x^{r+k-1}$$

$$y'' = \sum_{r=0}^\infty a_r (r+k)(r+k-1)\, x^{r+k-2}$$

$$xy' = \sum_{r=0}^{\infty} (r+k)\, a_r\, x^{r+k} \longrightarrow (1)$$

$$x^2y'' = \sum_{r=0}^{\infty} a_r \left((r+k)^2 - (r+k) \right) x^{r+k} \longrightarrow (2)$$

$$(x^2-n^2)y = (x^2-n^2) \sum_{r=0}^{\infty} a_r\, x^{k+r} \longrightarrow (3)$$

(1) + (2) + (3)

$$x^2y'' + xy' + (x^2-n^2)y = \sum_{r=0}^{\infty} a_r\, x^{r+k}\left((r+k)^2 - n^2\right)$$

$$\sum_{r=0}^{\infty} a_r\, x^{k+r+2} = 0$$

Comparing coefficients of lowest term (x^k)

$$a_0(k^2-n^2) = 0$$

$a_0 \neq 0$ $\qquad$ $k = \pm n$